A Bootstrap Approach for Generalized AutoCountour Testing Implications for VIX Forecast Densities

Gloria González-Rivera
University of California, Riverside

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J.H.G. Mazzeu, E. Ruiz and H. Veiga
Universidad Carlos III de Madrid

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Outline

- Motivation and Contribution
- Parametric Generalized AutoContouR (G-ACR) tests
- Bootstrap G-ACR tests (in-sample)
- Finite sample properties (in-sample)
- The VIX index.

Motivation

How to test for the correct specification of an h-step conditional predictive density when does not have a closed-form expression? e.g.

- the errors are non-Gaussian or have unknown distribution
- non-linear models in general
- linear models with conditional heteroscedasticity

Suppose $\{Y_t\}_{t=1}^{T}$ is the stochastic process of interest,

$$Y_t = g(Y^{t-1}, \theta_1) + \varepsilon_t \sigma_t$$

with $\varepsilon_t \rightarrow D(0, 1)$.

We wish to evaluate

$$Y_{t+h|t} \rightarrow \tilde{D}(\mu_{t+h|t}, \sigma^2_{t+h|t}; \theta_2) \text{ for } h > 1$$

Do we know $\tilde{D}(.)$?

Rich body of literature in density forecast. Methodology

- Diebold et al. (1998, IER)
- Diebold et al. (1999, RStat)
- Tay and Wallis (2000, JoF)
- Granger and Pesaran (2000, Stats&Finan.)
- Berkowitz (2001, JBES)
- Bai (2003, RStat)
- Hong and Li (2005, RFS)
- Corradi and Swanson (2006, Handbook)
- Amisano and Giacomini (2007, JBES)
- González-Rivera et al. (2011, JBES)
- Mitchell and Wallis (2011, JApE)
- González-Rivera and Yoldas (2012, IJF)
- Rossi and Sekhposyan (2013, JEcon)
- González-Rivera and Sun (2015, 2016, IJF)

Rich body of literature in density forecast. Applications

- Britton et al. (1998) fan charts of inflation
- Andersen et al. (2003, Econom.) volatility forecasting
- Clements et al. (2008, JEF) quantile forecasting exchange rates
- Corradi et al. (2009, JoE)) volatility forecasting
- Jore et al. (2010, JApE) combination of density forecasts
- Clark (2011, JBES) bayesian macro forecasting
- Maheu and McCurdy (2011, JoE) return distributions
- Baumeister and Kilian (2012, JBES) oil forecasting
- Hallam and Olmo (2013, JFinE) financial returns
- Alessi et al. (2014, JBES) macro forecasting
- Clark and Ravazzolo (2015, JApE) macro forecasting
- Nieto and Ruiz (2016, IJF) VaR forecasting

Contribution

1. Extension of the G-ACR tests computing the PITs from a bootstrapped conditional density
    - No need to assume a parametric density function. The only restrictions required on the error distribution are those to guarantee the consistency and asymptotic normality of the parameter estimator
    - Incorporate parameter uncertainty
    - The finite sample distribution is well approximated by standard asymptotic distributions
    - Easy implementation
    - Provides a graphical tool to spot potential misspecification
    - In-sample tests (dynamic misspecification) and out-of-sample tests (evaluation of density forecast)
2. Empirical application: testing the HAR model for the VIX

Let $\{Y_t\}_{t=1}^{T}$ be the univariate random process of interest with conditional density function $f_t\left(y_t|Y^{t-1}\right)$, where Y^{t-1} is the information set available at time $t-1$.

Based on the assumed conditional model, the researcher will construct density forecasts denoted by $g_t\left(y_t|Y_{t-1}\right)$. The sequence of PITs of $\{y_t\}_{t=1}^{T}$ w.r.t. $g_t\left(y_t|Y^{t-1}\right)$ is given by

$$u_t = \int_{-\infty}^{y_t} g_t\left(v_t|Y^{t-1}\right) dv_t$$

If $g_t\left(y_t|Y^{t-1}\right)$ coincides with the true $f_t\left(y_t|Y^{t-1}\right)$, then the sequence of PITs, $\{u_t\}_{t=1}^{T}$, is i.i.d. U(0,1).

[see Rosenblatt (1952, AMS) and Diebold et al. (1998, IER)]

Generalized AutoContouR (G-ACR)

González-Rivera, Senyuz, and Yoldas (2011 JBES)
González-Rivera and Sun (2015 IJF):

Define the set

$$\text{G-ACR}_{k,\alpha_i} = \left\{ \begin{array}{c} B(u_t, u_{t-k}) \subset \mathbb{R}^2 | 0 \leq u_t \leq \sqrt{\alpha_i} \text{ and } 0 \leq u_{t-k} \leq \sqrt{\alpha_i}, \\ \text{s.t. } u_t \times u_{t-k} \leq \alpha_i \end{array} \right\}$$

and the indicator

$$I_t^{k,\alpha_i} = 1\left((u_t, u_{t-k}) \in \text{G-ACR}_{k,\alpha_i}\right)$$

The sample proportion of PIT pairs within the G-ACR$_{k,\alpha_i}$ cube is given by

$$\widehat{\alpha}_{i,k} = \frac{\sum_{t=k+1}^{T} I_t^{k,\alpha_i}}{T-k}$$

The test

Under the null hypothesis, $H_0 : \{u_t\}_{t=1}^{T}$ *is* $i.i.d.U(0,1)$, the statistic

$$t_{k,\alpha_i} = \frac{\sqrt{T-k}\,(\widehat{\alpha}_{i,k} - \alpha_i)}{\sigma_{\alpha_i}}$$

where $\sigma^2_{\alpha_i} = \alpha_i(1-\alpha_i) + 2\alpha_i^{3/2}(1-\alpha_i^{1/2})$, is asymptotically N(0,1).

The adjustments of the asymptotic variance needed to take into account parameter uncertainty are model dependent and, consequently, difficult to calculate analytically. González-Rivera and Sun (2015) propose computing the adjusted σ_{α_i} by bootstrapping.

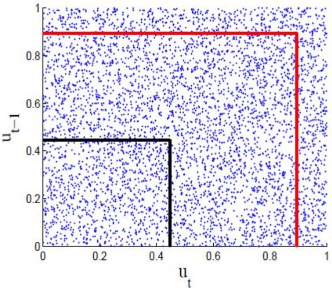
Some examples. I

Consider the following AR(1) model

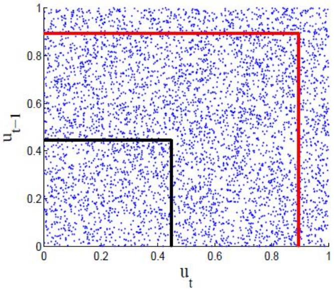
$$y_t = 0.5y_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim i.i.d.N(0,1).$$

Autocountours for $k = 1$ and $\alpha_i = 0.2$ (black) and 0.8 (red) and pairs (u_t, u_{t-1}) obtained after fitting an AR(1) and estimating the parameters by OLS.

Bootstrap density



Normal density



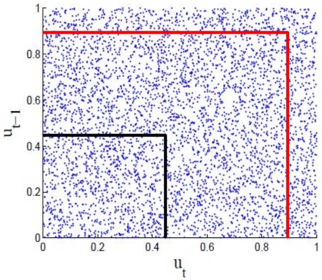
Some examples. II

Consider the following AR(1) model

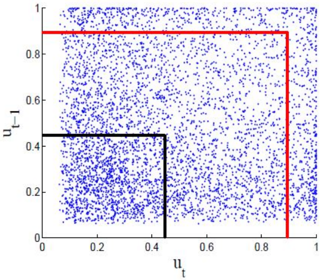
$$y_t = 0.95 y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim \text{independent centered and standardized } \chi^2_{(5)}.$$

Autocountours for $k = 1$ and $\alpha_i = 0.2$ (black) and 0.8 (red) and pairs (u_t, u_{t-1}) obtained after fitting an AR(1) and estimating the parameters by OLS.

Bootstrap density



Normal density



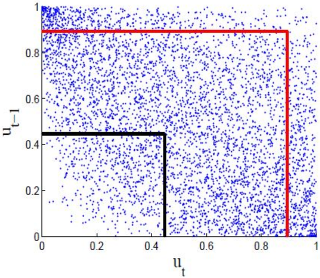
Some examples. III

Consider now the following DGP:

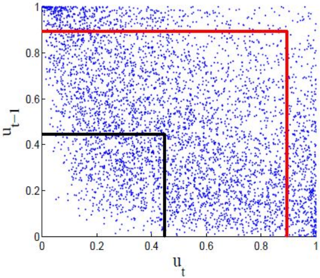
$$y_t = 0.3y_{t-1} + 0.6y_{t-2} + \varepsilon_t,\ \varepsilon_t \sim \text{independent centered and standardized } \chi^2_{(5)}$$

Autocountours for $k = 1$ and $\alpha_i = 0.2$ (black) and 0.8 (red) and pairs (u_t, u_{t-1}) obtained after **fitting an AR(1)** and estimating the parameters by OLS.

Bootstrap density



Normal density

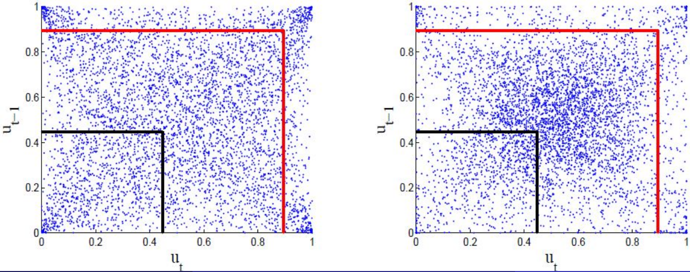


Consider now the following DGP:

$$
\begin{aligned}
y_t &= 0.5y_{t-1} + \varepsilon_t\sigma_t, \varepsilon_t \sim i.i.d.N(0,1). \\
\sigma_t^2 &= 0.05 + 0.5\varepsilon_{t-1}^2\sigma_{t-1}^2 + 0.45\sigma_{t-1}^2
\end{aligned}
$$

Autocountours for $k = 1$ and $\alpha_i = 0.2$ (black) and 0.8 (red) and pairs (u_t, u_{t-1}) obtained after **fitting an AR(1)** and estimating the parameters by OLS.

Bootstrap density | Normal density



Bootstrap autocountours (in-sample)

To focus the proposed procedure, consider the linear AR(1)-GARCH(1,1) model

$$y_t = \mu + \phi y_{t-1} + a_t$$

$$a_t = \varepsilon_t \sigma_t$$

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$$

where ε_t is i.i.d. with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = 1$ and distribution function F_ε . The model parameters guarantee stationarity and positivity and satisfy the conditions required for the parameter estimator to be consistent and asymptotically Normal.

The proposed procedure can be extended to alternative parametric specifications of the conditional mean and variance.

We propose implementing a combination of the residual bootstrap procedures proposed by Pascual et al. (2004, JTSA; 2006, CSDA) for ARMA and GARCH models.

Bootstrap autocountours (in-sample) (cont.)

1. Obtain residuals: Estimate the parameters by Gaussian QML and obtain the residuals $\widehat{\varepsilon}_t = \frac{y_t - \widehat{\mu} - \widehat{\phi} y_{t-1}}{\sqrt{\widehat{\omega} + \widehat{\alpha}\widehat{a}^2_{t-1} + \widehat{\beta}\widehat{\sigma}^2_{t-1}}}$. Denote by $\widehat{F}_{\widehat{\varepsilon}}$ the empirical distribution of the centered and scaled residuals.

2. Bootstrap replicates of parameter estimates: For $t = 3, ..., T$, obtain a boostrap replicate of y_t that mimics the dynamic dependence of the original series.

$$
\begin{aligned}
\sigma_t^{*2(b)} &= \widehat{\omega} + \widehat{\alpha} a_{t-1}^{*2(b)} + \widehat{\beta}\sigma_{t-1}^{*2(b)} \\
a_t^{*(b)} &= \sigma_t^{*(b)} \varepsilon_t^{*(b)} \\
y_t^{*(b)} &= \widehat{\mu} + \widehat{\phi} y_{t-1}^{*(b)} + a_t^{*(b)}
\end{aligned}
$$

where $\varepsilon_t^{*(b)}$ are random extractions with replacement from $\widehat{F}_{\widehat{\varepsilon}}$. Estimate the parameters by QML to obtain bootstrap estimates $\widehat{\mu}^{*(b)}$, $\widehat{\phi}^{*(b)}$, $\widehat{\omega}^{*(b)}$, $\widehat{\alpha}^{*(b)}$, and $\widehat{\beta}^{*(b)}$.

Bootstrap autocountours (in-sample) (cont.)

3. Obtain in-sample bootstrap one-step-ahead conditional densities: For $t = 3, \dots T$

$$\begin{aligned} \sigma_t^{**2(b)} &= \widehat{\omega}^{*(b)} + \widehat{\alpha}^{*(b)} \left(y_{t-1} - \widehat{\mu}^{*(b)} - \widehat{\phi}^{*(b)} y_{t-2} \right)^2 + \widehat{\beta}^{*(b)} \sigma_{t-1}^{**2(b)} \\ y_t^{**(b)} &= \widehat{\mu}^{*(b)} + \widehat{\phi}^{*(b)} y_{t-1} + \sigma_t^{**(b)} \varepsilon_t^{*(b)} \end{aligned}$$

4. Repeat steps 2 and 3 for $b = 1, ..., B^{(1)}$.

At each moment t, we obtain $B^{(1)}$ replicates of y_t conditional on $\{y_1, ..., y_{t-1}\}$, denoted by $y_t^{**(b)}$, $b = 1, ..., B^{(1)}$.

Then, in-sample PITs can be computed as follows

$$u_t = \frac{1}{B^{(1)}} \sum_{b=1}^{B^{(1)}} 1(y_t^{**(b)} < y_t)$$

The corresponding indicator I_t^{k,α_i} and sample proportion $\widehat{\alpha}_{i,k}$ can be computed as in the parametric G-ACR test.

The Boot-G-ACR test: Asymptotic distribution

The asymptotic distribution of the proposed statistics depends on the asymptotic validity of the residual bootstrap algorithm.

1. Pascual et al. (2004, JTSA) stablish the asymptotic validity of the bootstrap procedure in the context of ARMA models.

2. There is not a formal proof of the validity of the bootstrap in the context of GARCH models: Hidalgo and Zaffaroni (2007, JE) prove first order validity of the bootstrap to estimate the ARCH(∞) parameters.

If the bootstrap were valid for the estimation of the parameters, then one can stablish its validity for predictive densities using the arguments in Pascual et al. (2004, JTSA) and Reeves (2005, IJF).

Finite sample properties: Size (in-sample)

Consider the following AR(1) model

$$y_t = 0.95 y_{t-1} + \varepsilon_t$$

where ε_t is a centered and standardized independent $\chi^2_{(5)}$. Nominal size 5%.

α_i	0.05	0.3	0.5	0.7	0.95
			$T = 50$		
$\widehat{\alpha}_i$	0.058	0.307	0.504	0.705	0.950
size	0.046	0.015	0.016	0.013	0.004
			$T = 300$		
$\widehat{\alpha}_i$	0.052	0.303	0.502	0.701	0.949
size	0.036	0.034	0.032	0.031	0.018
			$T = 1000$		
$\widehat{\alpha}_i$	0.051	0.301	0.501	0.700	0.949
size	0.048	0.039	0.042	0.045	0.043

MC=1000; $B^{(1)}$=1000

Finite sample properties: Power (in-sample)

DGP is the AR(2) model

$$y_t = 0.3y_{t-1} + 0.6y_{t-2} + \varepsilon_t$$

where ε_t is i.i.d. Normal (0,1). The estimated model is an AR(1) model with the parameters estimated by OLS

α_i	0.05	0.3	0.5	0.7	0.95
			$T = 50$		
$\widehat{\alpha}_i$	0.001	0.095	0.294	0.553	0.902
power	0.000	0.588	0.389	0.11	0.046
			$T = 300$		
$\widehat{\alpha}_i$	0.002	0.128	0.344	0.600	0.926
power	1.000	1.000	1.000	0.995	0.36
$T = 1000$					
$\widehat{\alpha}_i$	0.002	0.135	0.355	0.609	0.930
power	1.000	1.000	1.000	1.000	0.948

MC=1000; $B^{(1)}$=1000

Finite sample properties: Power (in-sample)

DGP is the AR(1)-GARCH(1,1) and the estimated model is an AR(1) by OLS

$$
\begin{aligned}
y_t &= 0.5 y_{t-1} + \varepsilon_t \sigma_t; \ \varepsilon_t \sim i.i.d.N(0,1). \\
\sigma_t^2 &= 0.05 + 0.5 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + 0.45 \sigma_{t-1}^2
\end{aligned}
$$

	α_i	0.01	0.05	0.3	0.5	0.7	0.95
T=50	$\widehat{\alpha}_i$	0.018	0.063	0.312	0.522	0.719	0.953
	power	0.204	0.073	0.022	0.019	0.014	0.006
T=300	$\widehat{\alpha}_i$	0.023	0.066	0.308	0.517	0.718	0.953
	power	0.542	0.232	0.051	0.035	0.017	0.004
T=1000	$\widehat{\alpha}_i$	0.024	0.066	0.303	0.514	0.717	0.953
	power	0.929	0.494	0.048	0.066	0.105	0.070
T=5000	$\widehat{\alpha}_i$	0.024	0.066	0.302	0.512	0.716	0.954
	power	0.999	0.925	0.098	0.197	0.482	0.476

MC=1000; $B^{(1)}$=1000 (2000 for T=5000)

VIX Index

Daily market volatility index (VIX) from the Chicago Board Options Exchange (CBOE) observed daily from January 2, 1990 to January 15, 2013, $T = 5807$ (same sample period considered by Fernandez et al. (2014, JBF))

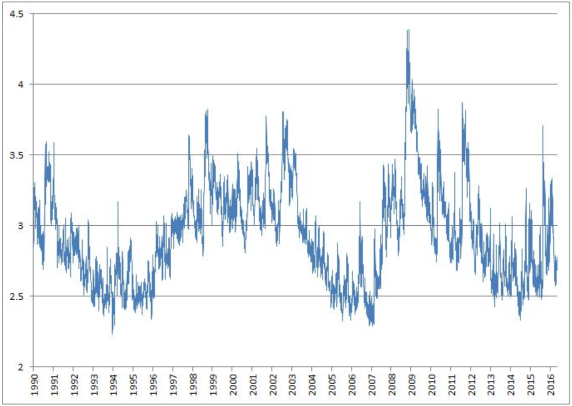


Figure 1: Daily log-VIX Index

ViX: Long memory

The series of log-VIX has skewness 0.539 and kurtosis 3.763. The sample autocorrelations of log-VIX and $(\text{log-VIX})^2$

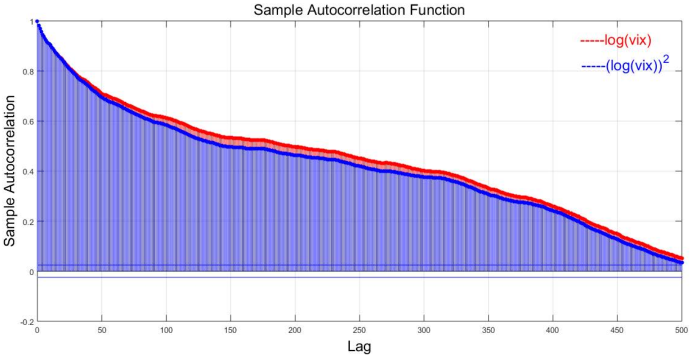


Figure 2: Autocorrelations

VIX: The HAR model

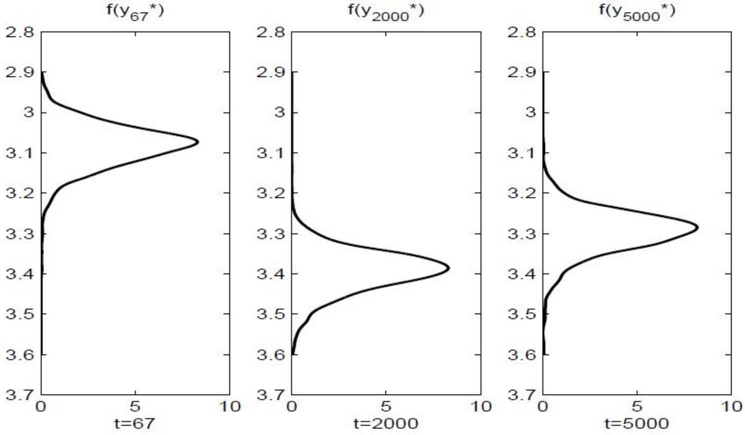
Heterogeneous AutoRegressive model of Corsi (2009, JFE):
1. capture memory persistence;
2. introduce the effect of low frequency volatilities towards high frequency volatility.

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_5 \overline{y}_{(t-1):5} + \phi_{10} \overline{y}_{(t-1):10} + \phi_{22} \overline{y}_{(t-1):22} + \phi_{66} \overline{y}_{(t-1):66} + \varepsilon_t$$

where $y_{t:i} = i^{-1} \sum_{j=0}^{i-1} y_{t-j}$ and ε_t is an independent white noise sequence.

Bootstrap conditional densities

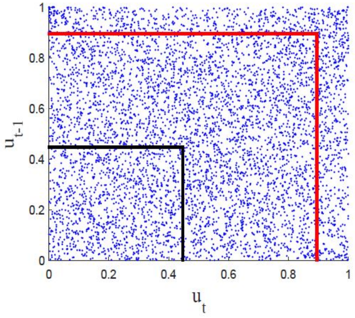
After estimating the HAR model, some in-sample one-step-ahead bootstrap conditional densities estimated in three different moments of time



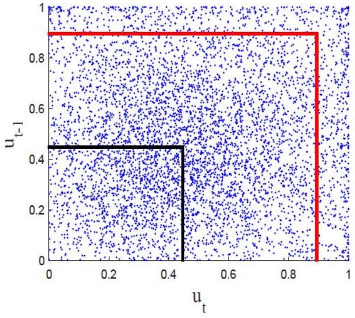
Generalized AutoContouR

After estimating HAR model, we plot the G-ACR using González-Rivera and Sun (2015) (assuming normality) and the Bootstrap G-ACR:

Bootstrap density



Normal density



In-sample G-ACR tests

t-statistics $|t_{\alpha_i,k=1}|$

α_i	0.01		0.5		0.95	
	Boot.	Normal	Boot.	Normal	Boot.	Normal
HAR	0.51	3.93**	4.08**	17.85**	0.15	7.56**
HAR-GARCH	1.85	5.41**	2.58*	12.88**	0.16	6.37**
HAR-GARCH-GJR	1.57	4.59**	2.18*	12.52**	0.49	6.46**

∗∗: 1% significance level

∗: 5% significance level

Conclusion

- Bootstrap G-ACR tests are easy to implement for evaluation of h-step-ahead predictive densities (in-sample and out-of-sample) that do not rely on any particular assumption on the error distribution and take into account parameter uncertainty
- Graphical device to point the potential misspecification of the estimated model. We can disentangle whether the misspecification comes from the assumed model or from the assumed density
- Good finite sample properties
- HAR model for VIX needs a heteroscedastic adjustment